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POLYNOMIAL PRESERVING MAPS ON CERTAIN JORDAN ALGEBRAS

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ABSTRACT

Let $\mathcal{B}$ and $\mathcal{Q}$ be associative algebras and let $\mathcal{S}$ be a Jordan subalgebra of $\mathcal{B}$. Let $f(x_1, \ldots, x_m)$ be a (noncommutative) multilinear polynomial such that $\mathcal{S}$ is closed under f. Let $\alpha\colon \mathcal{S} \to \mathcal{Q}$ be an f-homomorphism in the sense that it is a linear map preserving f. Under suitable conditions it is shown that α is essentially given by a ring homomorphism. An analogous theorem for f-derivations is also proved. The proofs rest heavily on results concerning functional identities and d-freeness.

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1. Introduction

A fundamental notion in the theory of associative algebras is that of an (anti)-homomorphism: if $\mathcal{B}$ and $\mathcal{Q}$ are associative algebras over a commutative ring $\mathcal{Z}$ then a $\mathcal{Z}$-module map $\sigma: \mathcal{B} \to \mathcal{Q}$ is a homomorphism (resp. anti-homomorphism) if $(st)^\sigma = s^\sigma t^\sigma$ (resp. $(st)^\sigma = t^\sigma s^\sigma$) for all $s,t \in \mathcal{B}$. The following general question arises. Let $f(x_1,\dots,x_m) \in \mathcal{Z}\langle\mathcal{X}\rangle$, $m > 1$, where $\mathcal{Z}\langle\mathcal{X}\rangle$ is the free $\mathcal{Z}$-algebra generated by an infinite set $\mathcal{X}$, let $\mathcal{S}$ be a $\mathcal{Z}$-submodule of $\mathcal{B}$ which is closed under f (that is, $f(\overline{s}_m) \in \mathcal{S}$ for all $\overline{s}_m = (s_1,\dots,s_m) \in \mathcal{S}^m$), and let $\alpha: \mathcal{S} \to \mathcal{Q}$ be an f-homomorphism in the sense that α is a $\mathcal{Z}$-module map such that $f(\overline{s}_m)^\alpha = f(\overline{s}_m^\alpha)$ for all $\overline{s}_m \in \mathcal{S}^m$ (here $\overline{s}_m^\alpha = (s_1^\alpha,\dots,s_m^\alpha)$). Must α be essentially expressed in terms of a homomorphism and/or an anti-homomorphism of $\langle\mathcal{S}\rangle$ into $\mathcal{Q}$ (here $\langle\mathcal{S}\rangle$ denotes the subalgebra generated by $\mathcal{S}$)? Of course in this generality the question is necessarily vague. Clearly one must impose some further conditions in order to have an affirmative solution. For instance, if $\mathcal{S} = \mathcal{Q}$ and f is a polynomial identity (PI) for $\mathcal{Q}$ then any map is an f-homomorphism. Extensive results have been obtained in the cases where $f = x_1x_2 - x_2x_1$ (the Lie case) and $f = x_1x_2 + x_2x_1$ (the Jordan case). It is natural to use these cases as a guide to what kind of conditions one might impose in general. The Lie and Jordan products arise naturally in algebras with (or without) involution, and so $\mathcal{S}$ is often taken to be the skew elements or the symmetric elements of an algebra with involution. The ring $\mathcal{Q}$ is often taken to be prime (or the image of α is assumed to have prime-like properties). In view of the earlier remark regarding PIs it is natural to assume that $\mathcal{Q}$ should not be PI of a too low degree. To put this another way, take $\mathcal{Q} = \mathcal{Q}_{mr}(\mathcal{A})$ to be the maximal ring of right quotients of a prime algebra $\mathcal{A}$ and let $\mathcal{R}$ be a subset of $\mathcal{Q}$. Then $\deg(\mathcal{R})$ is defined to be $\sup\{\deg(r) \mid r \in \mathcal{R}\}$ (possibly infinite), where $\deg(r)$ is the degree of the minimum polynomial over the extended centroid $\mathcal{C}$ of $\mathcal{A}$ satisfied by r. Then one imposes the condition that $\deg(\mathcal{A})$ is not too low.

Under such conditions the answers to our basic question above in the case of Lie and Jordan homomorphisms have been pretty well worked out.

In the Lie case the first general result was proved by Hua [21] in 1949. There followed a series of papers, notably [24] in 1975, in which the presence of orthogonal idempotents was assumed. The fundamental breakthrough (eliminating the requirement of idempotents) was made by Brešar [13] in 1993. It was the first paper in which functional identities were applied to the Lie map problems. The main idea of the proof can be easily described. If $\alpha: \mathcal{B} \to \mathcal{A}$ is a Lie isomorphism of prime rings, then clearly $[(\{x^{\alpha^{-1}}\}^2)^\alpha, x] = 0$ for all $x \in \mathcal{A}$. Under suitable assumptions, all biadditive maps $D: \mathcal{A}^2 \to \mathcal{A}$ satisfying $[D(x,x),x] = 0$ can be described [13]. Hence one obtains the form of $(x^2)^\alpha$ and finally that of $(xy)^\alpha$. Brešar's paper was followed by the general solution [10] in 1994 for the involution case where the functional identity $[(\{x^{\alpha^{-1}}\}^3)^\alpha, x] = 0$ was investigated (see also [16]). The use of power substitutions as the method of obtaining functional identities associated to Lie maps was not applicable to the description of Lie isomorphisms of Lie ideals (of skew elements) of prime rings (with involution) because a Lie ideal is not necessarily closed under the taking of powers of its elements. A new method of obtaining functional identities was developed in [7] where one described surjective Lie homomorphism onto Lie ideals of prime rings factorized by their centers. This method was also used in [2] where one investigated similar problems in the context of rings with involution. For a complete description of the Lie homomorphism results we refer the reader to [4].

The Jordan case has its roots in Herstein's 1956 paper [19] and also in papers [17, 18, 22] of Jacobson and Rickart, and Kaplansky. Further progress was made in 1967 under the assumption of orthogonal idempotents [23], but the main breakthrough came with Zelmanov's paper [27] in 1983 (see also McCrimmon's 1989 paper [26]).

Until recently relatively little had been achieved in case f was of degree higher than 2. In 1956 Herstein proved [19, Theorem K] that if $f = x^m$, $1 \in \mathcal{B}$, $\mathcal{Q}$ is prime with center $\mathcal{C}$, characteristic of $\mathcal{C}$ is either 0 or greater than m, and $\alpha: \mathcal{B} \to \mathcal{Q}$ is a surjective f-homomorphism, then there exists $\lambda \in \mathcal{C}$, $\lambda^{m-1} = 1$, such that $s^\alpha = \lambda s^\sigma$, σ an (anti-)homomorphism of $\mathcal{B}$ onto $\mathcal{Q}$. The assumption of $1 \in \mathcal{B}$ was subsequently removed in [15] in 1998. The general case of f-isomorphisms of prime algebras, where f is a multilinear polynomial of degree > 1, was investigated in [9] in 1999. In this paper a method of obtaining a functional identity associated to an f-isomorphism was found.

In his 1961 AMS Hour Talk Herstein posed the problem [20, p. 528, problem 1] for $f = x^m$ when $\mathcal{S}$ is either the symmetric elements or (for odd m) the skew elements of a simple algebra with involution. Some of these questions have been recently solved as corollaries to more general results. Namely, for $f(\overline{x}_m) \in \mathcal{Z}\langle\mathcal{X}\rangle$, $m > 1$, a multilinear polynomial of degree m with all nonzero coefficients invertible in $\mathcal{Z}$, f-homomorphisms have been determined in the following situations:

(I) $\mathcal{S}$ a Lie ideal of $\mathcal{B}$, $\mathcal{S}^\alpha$ a noncentral Lie ideal of a prime algebra $\mathcal{A}$, $\mathcal{Q} = \mathcal{Q}_{mr}(\mathcal{A})$, $\deg(\mathcal{A}) > \max\{2m, 7\}$ [8, Theorem 1.1];

(II) $\mathcal{B}$ with involution, $\mathcal{S}$ a Lie ideal of the skew elements of $\mathcal{B}$, $\mathcal{A}$ prime with

$*$ and skew elements $\mathcal{K}$, $\mathcal{Q} = \mathcal{Q}_{mr}(\mathcal{A})$, $\mathcal{S}^\alpha$ a noncentral Lie ideal of $\mathcal{K}$, $\deg(\mathcal{A}) > \max\{4m+2, 20\}$ [2, Theorem 1.5].

In view of the results (I) and (II) the situation which remains to be considered is the one in which $\mathcal{S}$ is the Jordan algebra of symmetric elements of an algebra $\mathcal{B}$ with involution. This is the principal topic of the present paper. The main results of this paper are Theorem 3.7 (in which f-homomorphisms are reduced to Jordan homomorphisms) and Theorem 3.9 (in which Jordan homomorphisms are reduced to ordinary homomorphisms). As a point of independent interest we remark that the conditions assumed in Theorem 3.9 enable us to bypass the Zelmanov approach to Jordan homomorphisms. As a corollary to Theorem 3.9 we have

THEOREM 1.1: *Let $\mathcal{B}$ be a $\mathcal{Z}$-algebra with involution and let $\mathcal{S}$ be the Jordan algebra of symmetric elements of $\mathcal{B}$. Let $f(\overline{x}_m)$, $m > 1$, be a proper (that is, all its nonzero coefficients are invertible in $\mathcal{Z}$) multilinear polynomial in $\mathcal{Z}\langle X\rangle$ such that $\mathcal{S}$ is closed under f. Let $\mathcal{A}$ be a prime $\mathcal{Z}$-algebra with involution $*$, with extended centroid $\mathcal{C}$, with* $\mathrm{char}(\mathcal{A}) \neq 2$ *and* $\mathcal{Q} = \mathcal{Q}_{mr}(\mathcal{A})$, *and suppose* $\deg(\mathcal{A}) > \max\{6m+1, 15\}$. *Let $\alpha\colon \mathcal{S} \to \mathcal{Q}$ be an f-homomorphism whose range $\mathcal{S}^\alpha$ contains the symmetric elements of some nonzero $*$-ideal $\mathcal{I}$ of $\mathcal{A}$. Further, suppose that at least one of the following two conditions holds:*

(i) *α is one-to-one and $\mathcal{S}$ does not satisfy a PI of degree $\le m+4$.*

(ii) *f is a Jordan polynomial.*

Then there exist $\lambda \in \mathcal{C}$, with $\lambda^{m-1} = 1$, $\psi\colon \mathcal{S} \to \mathcal{C}$ a $\mathcal{Z}$-module map, and a $\mathcal{Z}$-algebra homomorphism $\sigma\colon \langle\mathcal{S}\rangle \to \mathcal{Q}$ such that $s^\alpha = \lambda s^\sigma + \psi(s)$ for all $s \in \mathcal{S}$.

Furthermore, if $f^{(i)} = f(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_m) \neq 0$ for some i, then $\psi = 0$.

Another important notion in the theory of associative algebras is that of a derivation: if $\mathcal{B} \subseteq \mathcal{Q}$ are associative $\mathcal{Z}$-algebras then a $\mathcal{Z}$-module map $d\colon \mathcal{B} \to \mathcal{Q}$ is a derivation if $(st)^d = s^d t + st^d$ for all $s, t \in \mathcal{B}$. Let $\mathcal{Q}$ be a $\mathcal{Z}$-algebra, let $f(\overline{x}_m) \in \mathcal{Z}\langle\mathcal{X}\rangle$ be a multilinear polynomial, and let $\mathcal{S}$ be a $\mathcal{Z}$-submodule of $\mathcal{Q}$ which is closed under f. Then we define an f-derivation $\delta\colon \mathcal{S} \to \mathcal{Q}$ to be a $\mathcal{Z}$-module map such that

$$f(\overline{s}_m)^\delta = \sum_{i=1}^{m} f(s_1, \ldots, s_{i-1}, s_i^\delta, s_{i+1}, \ldots, s_m)$$

for all $\overline{s}_m \in \mathcal{S}^m$. In a sense the notion of an f-derivation δ is secondary to that of an f-homomorphism, since the map

$$s \mapsto \begin{pmatrix} s & s^\delta \\ 0 & s \end{pmatrix}$$

is an f-homomorphism (thus enabling one to be able to deduce f-derivation results from f-homomorphism results as it was first done in [3]). Thus we will not go into any details as we have done for f-homomorphisms; suffice it to say that there are indeed counterparts for f-derivations of all the above mentioned results for f-homomorphisms. In this paper the result we prove for f-derivations (Theorem 4.1) has as a corollary the following

THEOREM 1.2: *Let $\mathcal{A}$ be a prime $\mathcal{Z}$-algebra with involution, with extended centroid $\mathcal{C}$, with $\mathrm{char}(\mathcal{A}) \neq 2$ and $\mathcal{Q} = \mathcal{Q}_{mr}(\mathcal{A})$, and let $\mathcal{S}$ be the Jordan algebra of symmetric elements of $\mathcal{A}$. Let $f(\overline{x}_m)$, $m > 1$, be a proper multilinear polynomial in $\mathcal{Z}\langle\mathcal{X}\rangle$ such that $\mathcal{S}$ is closed under f. Suppose that $\deg(\mathcal{A}) > \max\{6m+1, 15\}$ and let $\delta\colon \mathcal{S} \to \mathcal{Q}$ be an f-derivation. Then there exist a derivation $d\colon \langle\mathcal{S}\rangle \to \mathcal{Q}$, $\lambda \in \mathcal{C}$, and a $\mathcal{Z}$-module map $\mu\colon \mathcal{S} \to \mathcal{Q}$ such that $s^d = s^\delta + \lambda s + \mu(s)$ for all $s \in \mathcal{S}$. Furthermore, we have:*

(a) *If the characteristic of $\mathcal{A}$ does not divide $m-1$, then $\lambda = 0$.*
(b) *If $f^{(i)} \neq 0$ for some i, then $\mu = 0$.*

The statements of the main theorems of this paper (Theorems 3.7, 3.9, and 4.1) involve the notion of d-free sets introduced in [5], and the proofs of these theorems depend heavily on several results concerning d-freeness obtained in [5, 6]. Therefore, in preparation for the main theorems, we shall give in section 2 a review of the definition of a d-free set and then gather together the statements of those results on d-freeness which are needed in sections 3 and 4. Since $\mathcal{S}$ is not closed under the Lie product, the method from [9] of obtaining a functional identity associated to an f-homomorphism (which was also used in [2, 8]) is not applicable to our situation, and so we develop a new one in section 3.

Throughout this paper we shall assume that all maps and structures respect $\mathcal{Z}$, even if $\mathcal{Z}$ is not always explicitly mentioned.

2. Functional identities and d-freeness

Before embarking on a description of the necessarily complicated functional identities required in this paper, we suggest that the uninitiated reader might find it helpful to look at the introductory account [14] of this subject by Brešar.

Let $\alpha\colon S \to \mathcal{Q}$ be a mapping of a set S into a unital algebra $\mathcal{Q}$; we denote the center of $\mathcal{Q}$ by C and the image of α by $\mathcal{R} = S^\alpha$. As a special case we have $\mathcal{S} = \mathcal{R}$ and $\alpha = \mathrm{id}_{\mathcal{R}}$. Let m be a positive integer. We will be considering several types of functions from $\mathcal{S}^m$ into $\mathcal{Q}$. The first two are the "basic building blocks" and the latter two are constructed from the first two.

(a) "Monomial" functions. Let $\mathcal{T} = \{x_1, \ldots, x_m\}$ be a finite set and let $\mathcal{M}_m$ denote the set of all formal multilinear monomials $M = x_{i_1} \cdots x_{i_k}$ of degree $k \leq m$. Each such M induces a function $M'\colon \mathcal{S}^m \to \mathcal{Q}$ by the rule $\overline{s}_m = (s_1, \ldots, s_m) \to s_{i_1}^\alpha s_{i_2}^\alpha \cdots s_{i_k}^\alpha = M^\alpha$.

(b) Arbitrary functions on $\mathcal{S}^n$, $n < m$. Given $B\colon \mathcal{S}^n \to \mathcal{Q}$ there are various ways that B induces a function of $\mathcal{S}^m$ into $\mathcal{Q}$. Namely, given $1 \leq j_1 < j_2 < \cdots < j_n \leq m$, then B induces a function $B'\colon \mathcal{S}^m \to \mathcal{Q}$ via $B'(s_1, \ldots, s_m) = B(s_{j_1}, \ldots, s_{j_n})$.

(c) Multilinear quasi-monomial functions. Let L^α be the monomial function determined by $L = x_{i_1} x_{i_2} \cdots x_{i_k}$, let $\lambda\colon \mathcal{S}^{m-k} \to \mathcal{C}$, and let λ_L be the function given by the rule $(s_1, \ldots, s_m) \to \lambda(s_{j_1}, s_{j_2}, \ldots, s_{j_{m-k}})$, where $j_t < j_{t+1}$ and $\{j_1, \ldots, j_{m-k}\}$, $\{i_1, \ldots, i_k\}$ is a partition of the set $\{1, 2, \ldots, m\}$. Then $\lambda_L L^\alpha$ is a function from $\mathcal{S}^m$ into $\mathcal{Q}$; it is called a multilinear quasi-monomial (function). A sum of such functions is called a multilinear quasi-polynomial (function) of degree $\leq m$.

(d) Functions of the form $M^\alpha B_{M,N} N^\alpha$. This (perhaps overly concise) notation is explained as follows. We write $M = x_{i_1} x_{i_2} \cdots x_{i_u}$ and $N = x_{j_1} x_{j_2} \cdots x_{j_v}$, where $\{i_1, \ldots, i_u\}$ and $\{j_1, \ldots, j_v\}$ are disjoint and $u + v \leq m$. Then $B_{M,N}\colon \mathcal{S}^n \to \mathcal{Q}$, $n = m - u - v$, is a function acting on the n-tuple $(s_{k_1}, \ldots, s_{k_n})$, where $\{i_1, \ldots, i_u\}$, $\{j_1, \ldots, j_v\}$, $\{k_1, \ldots, k_n\}$ is a partition of $\{1, 2, \ldots, m\}$ and $k_1 < k_2 < \cdots < k_n$.

We are now in a position to describe a certain kind of functional identity which will turn out to lie at the heart of this paper (we will forego attempting to give any general definition of "functional identity"). The identity we are about to write down (1) is, in more concise notation, the same identity being examined in [6, Theorem 2.6].

$$(1) \qquad \sum_{M,N} a_{M,N} M^\alpha B_{M,N} N^\alpha = \sum_L \lambda_L L^\alpha \quad \text{for all } (s_1, \ldots, s_m) \in \mathcal{S}^m.$$

Here $n < m$ is fixed, $a_{M,N} \in \mathcal{C}$, M, N ranges over $\mathcal{M}_m$ while following the constraints given in (d), and L ranges over $\mathcal{M}_m$ while conforming to (c). The idea here is that each $B_{M,N}$ is a function involving α in some way, and the hope is that (under suitable conditions on $\mathcal{R}$) the presence of the monomial functions M^α, N^α and the fact that the right hand side is a multilinear quasi-polynomial is somehow enough to force each $B_{M,N}$ to be a quasi-polynomial of degree $\leq n$.

As an example, let $\mathcal{S}$ be a ring and let $\alpha\colon \mathcal{S} \to \mathcal{Q}$ be a surjective Lie homomorphism. One linearizes the identity $[s^\alpha, (s^2)^\alpha] = 0$ to obtain

$$[s^\alpha, (tu + ut)^\alpha] + [t^\alpha, (su + us)^\alpha] + [u^\alpha, (st + ts)^\alpha] = 0 \tag{2}$$

for all $s, t, u \in \mathcal{S}$.

We note that (2) fits (1) if we take $m = 3$, $n = 2$, each $B(x, y) = (xy + yx)^\alpha$ and each $\lambda_L = 0$.

As a simpler example, let $\mathcal{S} = \mathcal{Q}$ be a prime ring and let α be a so-called centralizing map: $[s^\alpha, s] = \mu(s) \in \mathcal{C}$. Linearizing this identity we have $[s^\alpha, t] + [t^\alpha, s] = \lambda(s, t) \in \mathcal{C}$ for all $s, t \in \mathcal{S}$. This fits (1) with $m = 2$, $n = 1$, and each $B(x) = x^\alpha$. It is well known that the solution of this functional identity is the quasi-polynomial of degree 1: $s^\alpha = \rho s + \nu(s)$, $\rho \in \mathcal{C}$, $\nu\colon \mathcal{S} \to \mathcal{C}$ (see [12]).

Returning now to the functional identity (1), we note that all we are lacking now is a (reasonable) condition on $\mathcal{R}$ which will force each $B_{M,N}$ to be of the desired form. It turns out that one such condition is that $\mathcal{R}$ be a so-called d-free subset of $\mathcal{Q}$ (for appropriate d), and we now proceed to define this notion. The functional identities required for this definition are very special cases of (1). Let $\mathcal{R}$ be a subset of a unital algebra $\mathcal{Q}$. Let m be a positive integer, let $\mathcal{T} = \{x_1, \ldots, x_m\}$, and let $\mathcal{I}$ and $\mathcal{J}$ be subsets of $\mathcal{T}$. Furthermore let E_i, F_j, $i \in \mathcal{I}$, $j \in \mathcal{J}$, be functions of $\mathcal{R}^{m-1}$ into $\mathcal{Q}$. We consider the following FI's:

$$\sum_{i \in \mathcal{I}} E_i s_i + \sum_{j \in J} s_j F_j = 0 \quad \text{for all } \bar{s}_m \in \mathcal{R}^m; \tag{3}$$

$$\sum_{i \in \mathcal{I}} E_i s_i + \sum_{j \in J} s_j F_j \in \mathcal{C} \quad \text{for all } \bar{s}_m \in \mathcal{R}^m. \tag{4}$$

It is understood that E_i is the function acting on $(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_m)$, etc. One "obvious" solution of both (3) and (4) is

$$E_i = \sum_{i \neq j, j \in \mathcal{J}} s_j p_{ij} + \lambda_i; \quad F_j = -\sum_{i \neq j, i \in \mathcal{I}} p_{ij} s_i - \lambda_j, \tag{5}$$

where $p_{ij} = p_{ji}$ is a function: $\mathcal{R}^{m-2} \to \mathcal{Q}$ (it being understood that in (5) it acts on $(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_{j-1}, s_{j+1}, \ldots, s_m)$), and $\lambda_k\colon \mathcal{R}^{m-1} \to \mathcal{C}$ is a function acting on $(s_1, \ldots, s_{k-1}, s_{k+1}, \ldots, s_n)$ such that $\lambda_k = 0$ if k is not in $\mathcal{I} \cap \mathcal{J}$. We shall call (5) the standard solution of (3) and (4).

Definition 2.1: Let d be a positive integer and let $\mathcal{R}$ be a subset of a unital algebra $\mathcal{Q}$. Then $\mathcal{R}$ is a d-free subset of $\mathcal{Q}$ if the following conditions are satisfied:

1. For all positive integers m and subsets $\mathcal{I}$, $\mathcal{J}$ of $\{x_1, \ldots, x_m\}$ with $\max\{|\mathcal{I}|, |\mathcal{J}|\} \le d$, (3) implies (5).

2. For all positive integers m and subsets $\mathcal{I}$, $\mathcal{J}$ of $\{x_1, \dots, x_m\}$ with $\max\{|\mathcal{I}|, |\mathcal{J}|\} < d$, (4) implies (5).

Clearly if $\mathcal{R}$ is d-free and $d' \leq d$ then $\mathcal{R}$ is also d'-free. If $\mathcal{R}$ is d-free, then any identity of the form (3) with $m \leq d$ has only the standard solution (5) and any identity of the form (4) with $m < d$ has only the standard solution (5).

We now list the theorems on d-freeness which will be needed in this paper. Our first theorem shows that if $\mathcal{A}$ is a prime algebra and $\mathcal{Q} = \mathcal{Q}_{mr}(\mathcal{A})$ then (put rather loosely) various important subsets of $\mathcal{A}$ will be d-free if $\deg(\mathcal{A})$ is sufficiently high. If $\mathcal{A}$ is algebraic of bounded degree over $\mathcal{C}$ then $\deg(\mathcal{A})$ is defined to be the least such bound n; otherwise $\deg(\mathcal{A}) = \infty$. There are two equivalent conditions to $n = \deg(\mathcal{A})$:

1. $\dim(\mathcal{AC} : \mathcal{C}) = n^2$ and
2. $2n$ is the minimum degree of a polynomial identity for $\mathcal{A}$.

THEOREM 2.2: *Let $\mathcal{A}$ be a prime algebra with $\mathcal{Q} = \mathcal{Q}_{mr}(\mathcal{A})$ and let $n = \deg(\mathcal{A})$.*

(a) *If $\mathcal{I}$ is a nonzero ideal of $\mathcal{A}$, then $\deg(\mathcal{I}) = n$ and $\mathcal{I}$ is n-free in $\mathcal{Q}$ [5, Lemma 2.1 and Corollary 2.10].*

(b) *If $\mathcal{R}$ is a noncentral Lie ideal of $\mathcal{A}$ and $n > d$, then $\mathcal{R}$ is d-free in $\mathcal{Q}$ [5, Theorem 2.20].*

(c) *Suppose furthermore that $\mathcal{A}$ has an involution, with $\mathcal{K}$ denoting the skew elements and $\mathcal{S}$ the symmetric elements.*

(c1) *If $\mathrm{char}(\mathcal{A}) \neq 2$, $\mathcal{R}$ is a noncentral Lie ideal of $\mathcal{K}$ and $n > 2d + 2$, then $\mathcal{R}$ is d-free in $\mathcal{Q}$ [1, Theorem 1.1].*

(c2) *If $\mathrm{char}(\mathcal{A}) \neq 2$, $n > 2d + 1$ then $\mathcal{S}$ is d-free in $\mathcal{Q}$ [5, Theorem 2.4].*

A consequence of Theorem 2.2 is that if any one of the sets $\mathcal{A}$, $\mathcal{I}$, $\mathcal{K}$, $\mathcal{S}$, $\mathcal{R}$ listed above is not d-free for some d, then $\mathcal{A}$ must be a PI algebra.

Next we have a general theorem about d-freeness.

THEOREM 2.3: *Let $\mathcal{R}$ be a subset of a unital ring $\mathcal{Q}$, and let $\mathcal{C}$ be the center of $\mathcal{Q}$.*

(a) *If $\mathcal{R}$ is d-free in $\mathcal{Q}$ and $\mathcal{T}$ is a subset of $\mathcal{Q}$ containing $\mathcal{R}$ then $\mathcal{T}$ is d-free in $\mathcal{Q}$ [5, Corollary 2.9].*

(b) *If λ is an invertible element of $\mathcal{C}$, then $\mathcal{R}$ is d-free if and only if $\lambda\mathcal{R}$ is d-free [5, Remark 2.11(c)].*

In preparation for the next result let $\mathcal{Q}$ be a unital ring with center $\mathcal{C}$. We define $\widehat{\mathcal{Q}}$ to be the direct sum $\mathcal{Q} \oplus \mathcal{Q}$ with multiplication given by $(x, y)(u, v) = (xu, xv + yu)$. Note that the center of $\widehat{\mathcal{Q}}$ is $\widehat{\mathcal{C}} = \mathcal{C} \oplus \mathcal{C}$. The following theorem is very useful in making the transition from f-derivations to f-homomorphisms.

THEOREM 2.4: *If $\mathcal{R}$ is a d-free subset of a unital ring $\mathcal{Q}$ and δ: $\mathcal{R} \to \mathcal{Q}$ is any (set-theoretic) map, then $\mathcal{R}' = \{(x, x^\delta) \mid x \in \mathcal{R}\}$ is a d-free subset of $\widehat{\mathcal{Q}}$ [3, Theorem 2.1].*

We now return to the discussion of the more complicated functional identities described at the beginning of this section, and we refer the reader to those passages explaining various terminologies and conditions. Thus we are now assuming that α: $\mathcal{S} \to \mathcal{Q}$ is a mapping from a set $\mathcal{S}$ into a unital algebra $\mathcal{Q}$, with $\mathcal{R} = \mathcal{S}^\alpha$ and center of $\mathcal{Q} = \mathcal{C}$. In the next two theorems (which are critical to this paper) m will be a fixed positive integer and the functional identities under discussion take place in the setting of functions from $\mathcal{S}^m$ into $\mathcal{Q}$.

THEOREM 2.5 ([6, Theorem 1.1]): *Suppose $\mathcal{S}$ satisfies a multilinear quasi-polynomial identity*

$$\sum_{L \in \mathcal{M}_m} \lambda_L L^\alpha = 0 \quad \text{for all } \overline{s}_m \in \mathcal{S}^m.$$

If $\mathcal{R}$ is $(m+1)$-free in $\mathcal{Q}$ then each "coefficient" $\lambda_L = 0$.

The next theorem treats the functional identity (1) described earlier this section; under assumptions of d-freeness and an additional technical condition it is shown that the functions involved are (as hoped) indeed multilinear quasi-polynomials.

THEOREM 2.6 ([6, Theorem 2.6]): *Let $n < m$ be fixed, and suppose $\mathcal{S}$ satisfies the functional identity (1), that is*

$$\sum_{M,N} a_{M,N} M^\alpha B_{M,N} N^\alpha = \sum_{L} \lambda_L L^\alpha$$

for all $\overline{s}_m \in \mathcal{S}^m$ (where various notations and conditions have been explained earlier in (a)–(d)). Furthermore, assume the following two conditions:

1. *$\mathcal{R}$ is $(m+1)$-free in $\mathcal{Q}$.*
2. *Let u be the smallest $\deg(M)$ for which $a_{M,N} B_{M,N} \neq 0$ for some N. Suppose that for every pair M, N such that $a_{M,N} B_{M,N} \neq 0$ there exist a pair U, V with $\deg(U) = u$, $B_{U,V} = B_{M,N}$, and $a_{U,V}$ is invertible in $\mathcal{C}$.*

Then each $B_{M,N}$ is a multilinear quasi-polynomial of degree $\leq n$, that is,

$$B_{M,N} = \sum_{K \in \mathcal{M}_n} \mu_{M,N,K} K^\alpha \quad \text{for all } \overline{s}_n \in \mathcal{S}^n.$$

Moreover, if $\mathcal{Q}$ is an algebra over a commutative ring $\mathcal{Z}$, $\mathcal{S}$ is a $\mathcal{Z}$-module and each $B_{M,N}$ is a multilinear map, then all $\mu_{M,N,K}$ are multilinear maps.

Condition 2 of this theorem in particular means that if (1) contains a nonzero summand $a_{1,N_1}B_{1,N_1}N_1^\alpha$ (the case when $u=0$), then for any function $B_{M,N}$ involved in (1) we assume that (1) contains a summand $a_{1,N_2}B_{1,N_2}N_2^\alpha$ where $B_{1,N_2}=B_{M,N}$ and a_{1,N_2} is invertible in $\mathcal{C}$.

The final theorem we present in this section will be crucial to the proof of Theorem 3.9 and is a special case of [7, Theorem 2.7].

THEOREM 2.7: *Let $\mathcal{U}$ be a Lie ideal of an associative $\mathcal{Z}$-algebra $\mathcal{B}$ and let $\mathcal{Q}$ be a unital associative $\mathcal{Z}$-algebra with center $\mathcal{C}$ such that $\frac{1}{2}\in\mathcal{C}$, $\mathcal{C}$ is a $\mathcal{C}$-direct summand of $\mathcal{Q}$, and 1 is the only nonzero idempotent in $\mathcal{C}$. Let $\gamma\colon\mathcal{U}\to\overline{\mathcal{Q}}=\mathcal{Q}/\mathcal{C}$ be a Lie homomorphism such that $\mathcal{U}^\gamma=\overline{\mathcal{R}}$ where $\mathcal{R}$ is a 7-free subset of $\mathcal{Q}$. Then there exists a $\mathcal{Z}$-map $\sigma\colon\langle\mathcal{U}\rangle\to\langle\mathcal{R}\rangle\mathcal{C}+\mathcal{C}$ such that $x^\gamma=\overline{x^\sigma}$ for all $x\in\mathcal{U}$ and σ is either a homomorphism or the negative of an antihomomorphism.*

3. f-homomorphisms

Let $\mathcal{B}$ be an associative ring, let $\mathcal{S}$ be a Jordan subring of $\mathcal{B}$ and let $\mathcal{Q}$ be an associative ring with 1 and with center $\mathcal{C}$. Let $\frac{1}{2}\in\mathcal{C}$. Moreover, let $\mathcal{B}$ and $\mathcal{Q}$ be algebras over a commutative unital ring $\mathcal{Z}$ with $\mathcal{Z}\mathcal{S}=\mathcal{S}$, let $f(x_1,\ldots,x_m)\in\mathcal{Z}\langle\mathcal{X}\rangle$, $m\geq 2$ be a proper multilinear polynomial (that is, a multilinear polynomial all of whose nonzero coefficients are invertible in $\mathcal{Z}$) of degree m, and let $\mathcal{S}$ be closed under f, that is, $f(\overline{u}_m)\in\mathcal{S}$ for all $\overline{u}_m=(u_1,\ldots,u_m)\in\mathcal{S}^m$. Let $\alpha\colon\mathcal{S}\to\mathcal{Q}$ be an f-homomorphism, that is, α is a $\mathcal{Z}$-module map satisfying

$$f(\overline{u}_m)^\alpha=f(\overline{u}_m^\alpha)\quad\text{for all }\overline{u}_m\in\mathcal{S}^m. \tag{6}$$

Our first main goal (Theorem 3.7) in this section is to show, under appropriate conditions, that α is essentially determined by a Jordan homomorphism $\beta\colon\mathcal{S}\to\mathcal{Q}$. The overriding condition which is at the core of our arguments is that $\mathcal{S}^\alpha$ is a $3m$-free subset of $\mathcal{Q}$ and we thereby impose this condition now. There are a variety of results, under conditions notably based on the use of the so-called Zelmanov polynomial [25, 26], which asserts that Jordan homomorphisms are restrictions of homomorphisms or antihomomorphisms. In our final goal of this section (Theorem 3.9), however, we choose to steer clear of imposing the additional conditions required for the Zelmanov-based results. Instead we shall focus on a (generalization of the) situation in which $\mathcal{B}$ is a ring with involution and $\mathcal{S}$ is the Jordan ring of symmetric elements of $\mathcal{B}$; the passage from Jordan homomorphisms to homomorphisms will, rather surprisingly, follow as a corollary to a result on Lie homomorphisms proved by Beidar and Chebotar [7, Theorem 2.7].

Now we shall give the reader a road map motivating the sequence of steps leading up to Theorem 3.7. A reasonable (and ambitious!) first goal is to show that α preserves (or at least comes close to preserving) the product

$$[[w,v],u] = (u \circ v) \circ w - (u \circ w) \circ v \in \mathcal{S} \quad \text{for all } u,v,w \in \mathcal{S}, \tag{7}$$

where $[u,v] = uv - vu$ and $u \circ v = uv + vu$. It is therefore natural to consider the function $B\colon \mathcal{S}^3 \to \mathcal{Q}$ given by $B(u,v,w) = [[u,v],w]^\alpha$. It is easy to see that B satisfies each of conditions (9), (10), (11) given in Proposition 3.1. Indeed, (10) is just the Jacobi identity for the Lie product $[x,y]$, (11) is just the anticommutativity of $[x,y]$ and (9) stems from the well-known derivation formula

$$[[u,v], w_1w_2\cdots w_m] = \sum_{i=1}^{m} w_1\cdots w_{i-1}[[u,v],w_i]w_{i+1}\cdots w_m.$$

Proposition 3.1, whose proof is quite formal and relies heavily on d-freeness theorems stated in Section 2, accomplishes the major step of getting (Corollary 3.2) to

$$[[u,v],w]^\alpha = \lambda_1[[u^\alpha,v^\alpha],w^\alpha] + \omega(u,v,w), \tag{8}$$

$\lambda_1 \in \mathcal{C}$, $\omega\colon \mathcal{S}^3 \to \mathcal{C}$. It is imperative that λ_1 at least be invertible in $\mathcal{C}$ and to this end it is assumed that either of two fairly natural conditions hold; these are given as conditions (i) and (ii) just preceding Lemma 3.4, which asserts that under (i) or (ii), λ_1 is indeed invertible. Lemma 3.3 is needed in the proof of Lemma 3.4 in the case of condition (ii). Making use of λ_1 being invertible, Lemma 3.5 then shows that α acts in a Jordan-like manner on the Jordan product $u\circ v = uv+vu$, $u,v \in \mathcal{S}$, and Lemma 3.6 then defines a Jordan homomorphism $\beta\colon \mathcal{S} \to \mathcal{Q}$ such that $u^\beta = \lambda u^\alpha + \mu(u)$, $u \in \mathcal{S}$.

We start with the following proposition, noting that we do not need to assume $\mathcal{S}$ is a Jordan subring, only that $\mathcal{S}$ is an additive subgroup closed under f and $[[\mathcal{S},\mathcal{S}],\mathcal{S}] \subseteq \mathcal{S}$.

PROPOSITION 3.1: *Suppose that a map $B\colon \mathcal{S}^3 \to \mathcal{Q}$ satisfies*

$$B(u,v,f(\overline{w}_m)) = \sum_{i=1}^{m} f(w_1^\alpha,\ldots,w_{i-1}^\alpha,B(u,v,w_i),w_{i+1}^\alpha,\ldots,w_m^\alpha) \tag{9}$$

for all $u,v,w_1,\ldots,w_m \in \mathcal{S}$,

$$B(u,v,w) + B(v,w,u) + B(w,u,v) = 0 \tag{10}$$

for all $u, v, w \in \mathcal{S}$, and

$$(11) \qquad B(u,v,w) + B(v,u,w) = 0 \quad \textit{for all } u,v,w \in \mathcal{S}.$$

If $\mathcal{S}^\alpha$ *is a* $(3m)$*-free subset of* $\mathcal{Q}$, *then there exist* $\lambda_1 \in \mathcal{C}$ *and a map* $\omega \colon \mathcal{S}^3 \to \mathcal{C}$ *such that*

$$B(u,v,w) = \lambda_1[[u^\alpha, v^\alpha], w^\alpha] + \omega(u,v,w) \quad \textit{for all } u,v,w \in \mathcal{S}.$$

Proof: By (10) we have

$$B(f(\overline{u}_m), f(\overline{v}_m), f(\overline{w}_m)) + B(f(\overline{v}_m), f(\overline{w}_m), f(\overline{u}_m)) + B(f(\overline{w}_m), f(\overline{u}_m), f(\overline{v}_m)) = 0,$$

which can be, according to (9), written in the form

$$\sum_{i=1}^m f(w_1^\alpha, \ldots, w_{i-1}^\alpha, B(f(\overline{u}_m), f(\overline{v}_m), w_i), w_{i+1}^\alpha, \ldots, w_m^\alpha)$$
$$+\sum_{i=1}^m f(u_1^\alpha, \ldots, u_{i-1}^\alpha, B(f(\overline{v}_m), f(\overline{w}_m), u_i), u_{i+1}^\alpha, \ldots, u_m^\alpha)$$
$$+\sum_{i=1}^m f(v_1^\alpha, \ldots, v_{i-1}^\alpha, B(f(\overline{w}_m), f(\overline{u}_m), v_i), v_{i+1}^\alpha, \ldots, v_m^\alpha) = 0$$

for all $\overline{u}_m, \overline{v}_m, \overline{w}_m \in \mathcal{S}^m$. By Theorem 2.6 with $3m$ replacing m and $2m+1$ replacing n, there exist $\mu_K \colon \mathcal{S}^{2m+1-\deg(K)} \to \mathcal{C}$, $K \in \mathcal{M}_{2m+1}$, such that

$$B(f(\overline{u}_m), f(\overline{v}_m), w) = \sum_{K \in \mathcal{M}_{2m+1}} \mu_K(\overline{u}_m, \overline{v}_m, w) K(\overline{u}_m^\alpha, \overline{v}_m^\alpha, w^\alpha)$$

for all $\overline{u}_m, \overline{v}_m \in \mathcal{S}^m, w \in \mathcal{S}$. Next, as a special case of (10) we have

$$B(f(\overline{u}_m), f(\overline{v}_m), w) + B(f(\overline{v}_m), w, f(\overline{u}_m)) + B(w, f(\overline{u}_m), f(\overline{v}_m)) = 0,$$

which can be written as

$$\sum_{K \in \mathcal{M}_{2m+1}} \mu_K(\overline{u}_m, \overline{v}_m, w) K(\overline{u}_m^\alpha, \overline{v}_m^\alpha, w^\alpha)$$
$$+\sum_{i=1}^m f(u_1^\alpha, \ldots, u_{i-1}^\alpha, B(f(\overline{v}_m), w, u_i), u_{i+1}^\alpha, \ldots, u_m^\alpha)$$
$$+\sum_{i=1}^m f(v_1^\alpha, \ldots, v_{i-1}^\alpha, B(w, f(\overline{u}_m), v_i), v_{i+1}^\alpha, \ldots, v_m^\alpha) = 0,$$

for all $\overline{u}_m, \overline{v}_m \in \mathcal{S}^m$, $w \in \mathcal{S}$. By (11) and Theorem 2.6 with $2m+1$ replacing m and $m+2$ replacing n, there exist $\nu_K\colon \mathcal{S}^{m+2-\deg(K)} \to \mathcal{C}$, $K \in \mathcal{M}_{m+2}$, such that

$$B(f(\overline{u}_m), v, w) = \sum_{K \in \mathcal{M}_{m+2}} \nu_K(\overline{u}_m, v, w) K(\overline{u}_m^\alpha, v^\alpha, w^\alpha)$$

for all $\overline{u}_m \in \mathcal{S}^m$, $v, w \in \mathcal{S}$. Next, (10) implies

$$B(f(\overline{u}_m), v, w) + B(v, w, f(\overline{u}_m)) + B(w, f(\overline{u}_m), v) = 0,$$

which can be, in view of (11), written as

$$\sum_{K \in \mathcal{M}_{m+2}} \omega_K(\overline{u}_m, v, w) K(\overline{u}_m^\alpha, v^\alpha, w^\alpha) + \sum_{i=1}^{m} f(u_1^\alpha, \dots, u_{i-1}^\alpha, B(v, w, u_i), u_{i+1}^\alpha, \dots, u_m^\alpha) = 0$$

for all $\overline{u}_m \in \mathcal{S}^m$, $v, w \in \mathcal{S}$, where ω_K is the sum of several ν_L. By Theorem 2.6 there exist $\lambda_i \in \mathcal{C}$, $i = 1, \dots, 6$, $\mu_i\colon \mathcal{S} \to \mathcal{C}$, $i = 1, \dots, 6$ $\nu_i\colon \mathcal{S}^2 \to \mathcal{C}$, $i = 1, 2, 3$ and $\omega\colon \mathcal{S}^3 \to \mathcal{C}$ such that

$$\begin{aligned} B(u, v, w) = &\lambda_1 u^\alpha v^\alpha w^\alpha + \lambda_2 u^\alpha w^\alpha v^\alpha + \lambda_3 v^\alpha u^\alpha w^\alpha \\ &+ \lambda_4 v^\alpha w^\alpha u^\alpha + \lambda_5 w^\alpha u^\alpha v^\alpha + \lambda_6 w^\alpha v^\alpha u^\alpha \\ &+ \nu_1(u) v^\alpha w^\alpha + \nu_2(u) w^\alpha v^\alpha + \nu_3(v) u^\alpha w^\alpha \\ &+ \nu_4(v) w^\alpha u^\alpha + \nu_5(w) u^\alpha v^\alpha + \nu_6(w) v^\alpha u^\alpha \\ &+ \mu_1(u, v) w^\alpha + \mu_2(u, w) v^\alpha + \mu_3(v, w) u^\alpha + \omega(u, v, w) \end{aligned}$$

for all $u, v, w \in \mathcal{S}$. Applying Theorem 2.5 we see that (11) yields

$$\begin{aligned} &\lambda_1 + \lambda_3 = \lambda_2 + \lambda_4 = \lambda_5 + \lambda_6 = 0, \\ &\nu_1(u) + \nu_3(u) = \nu_2(u) + \nu_4(u) = \nu_5(u) + \nu_6(u) = 0, \\ &\mu_1(u, v) + \mu_1(v, u) = \mu_2(u, v) + \mu_3(u, v) = 0, \end{aligned}$$

for all $u, v \in \mathcal{S}$. Therefore,

$$\begin{aligned} B(u, v, f(\overline{w}_m)) = &\lambda_1 [u^\alpha, v^\alpha] f(\overline{w}_m^\alpha) + \lambda_2 u^\alpha f(\overline{w}_m^\alpha) v^\alpha \\ &- \lambda_2 v^\alpha f(\overline{w}_m^\alpha) u^\alpha + \lambda_5 f(\overline{w}_m^\alpha)[u^\alpha, v^\alpha] \\ &+ \nu_1(u) v^\alpha f(\overline{w}_m^\alpha) + \nu_2(u) f(\overline{w}_m^\alpha) v^\alpha \\ &- \nu_1(v) u^\alpha f(\overline{w}_m^\alpha) - \nu_2(v) f(\overline{w}_m^\alpha) u^\alpha \\ &+ \nu_5(f(\overline{w}_m))[u^\alpha, v^\alpha] + \mu_1(u, v) f(\overline{w}_m^\alpha) \\ &+ \mu_2(u, f(\overline{w}_m)) v^\alpha - \mu_2(v, f(\overline{w}_m)) u^\alpha \\ &+ \omega(u, v, f(\overline{w}_m)), \end{aligned}$$

while on the other hand,

$$
\begin{aligned}
B(u,v,f(\overline{w}_m)) &= \sum_{i=1}^{m} f(w_1^\alpha,\dots,w_{i-1}^\alpha,B(u,v,w_i),w_{i+1}^\alpha,\dots,w_m^\alpha)\\
&= \sum_{i=1}^{m} f(w_1^\alpha,\dots,w_{i-1}^\alpha,\{\lambda_1[u^\alpha,v^\alpha]w_i^\alpha + \lambda_2 u^\alpha w_i^\alpha v^\alpha - \lambda_2 v^\alpha w_i^\alpha u^\alpha\\
&\quad + \lambda_5 w_i^\alpha[u^\alpha,v^\alpha] + \nu_1(u)v^\alpha w_i^\alpha + \nu_2(u)w_i^\alpha v^\alpha - \nu_1(v)u^\alpha w_i^\alpha\\
&\quad - \nu_2(v)w_i^\alpha u^\alpha + \nu_5(w_i)[u^\alpha,v^\alpha] + \mu_1(u,v)w_i^\alpha + \mu_2(u,w_i)v^\alpha\\
&\quad - \mu_2(v,w_i)u^\alpha + \omega(u,v,w_i)\},w_{i+1}^\alpha,\dots,w_m^\alpha).
\end{aligned}
$$

Comparing we get

$$
\begin{aligned}
&\lambda_1[u^\alpha,v^\alpha]f(\overline{w}_m^\alpha) + \lambda_2 u^\alpha f(\overline{w}_m^\alpha)v^\alpha - \lambda_2 v^\alpha f(\overline{w}_m^\alpha)u^\alpha\\
&+ \lambda_5 f(\overline{w}_m^\alpha)[u^\alpha,v^\alpha] + \nu_1(u)v^\alpha f(\overline{w}_m^\alpha) + \nu_2(u)f(\overline{w}_m^\alpha)v^\alpha\\
&- \nu_1(v)u^\alpha f(\overline{w}_m^\alpha) - \nu_2(v)f(\overline{w}_m^\alpha)u^\alpha + \nu_5(f(\overline{w}_m))[u^\alpha,v^\alpha]\\
&+ \mu_1(u,v)f(\overline{w}_m^\alpha) + \mu_2(u,f(\overline{w}_m))v^\alpha - \mu_2(v,f(\overline{w}_m))u^\alpha\\
&+ \omega(u,v,f(\overline{w}_m)) = \sum_{i=1}^{m} f(w_1^\alpha,\dots,w_{i-1}^\alpha,\{\lambda_1[u^\alpha,v^\alpha]w_i^\alpha\\
&+ \lambda_2 u^\alpha w_i^\alpha v^\alpha - \lambda_2 v^\alpha w_i^\alpha u^\alpha + \lambda_5 w_i^\alpha[u^\alpha,v^\alpha]\\
&+ \nu_1(u)v^\alpha w_i^\alpha + \nu_2(u)w_i^\alpha v^\alpha - \nu_1(v)u^\alpha w_i^\alpha\\
&- \nu_2(v)w_i^\alpha u^\alpha + \nu_5(w_i)[u^\alpha,v^\alpha] + \mu_1(u,v)w_i^\alpha\\
&+ \mu_2(u,w_i)v^\alpha - \mu_2(v,w_i)u^\alpha + \omega(u,v,w_i)\},w_{i+1}^\alpha,\dots,w_m^\alpha)
\end{aligned}
$$

for all $u,v,w_1,\dots,w_m\in\mathcal{S}$. By Theorem 2.5 we have, in particular,

$$
\begin{aligned}
(12)\quad &\lambda_1 u^\alpha v^\alpha f(\overline{w}_m^\alpha) + \lambda_2 u^\alpha f(\overline{w}_m^\alpha)v^\alpha - \lambda_1 v^\alpha u^\alpha f(\overline{w}_m^\alpha)\\
&- \lambda_2 v^\alpha f(\overline{w}_m^\alpha)u^\alpha + \lambda_5 f(\overline{w}_m^\alpha)u^\alpha v^\alpha - \lambda_5 f(\overline{w}_m^\alpha)v^\alpha u^\alpha\\
&= \sum_{i=1}^{m} f(w_1^\alpha,\dots,w_{i-1}^\alpha,\{\lambda_1 u^\alpha v^\alpha w_i^\alpha + \lambda_2 u^\alpha w_i^\alpha v^\alpha - \lambda_1 v^\alpha u^\alpha w_i^\alpha\\
&- \lambda_2 v^\alpha w_i^\alpha u^\alpha + \lambda_5 w_i^\alpha u^\alpha v^\alpha - \lambda_5 w_i^\alpha v^\alpha u^\alpha\},w_{i+1}^\alpha,\dots,w_m^\alpha),
\end{aligned}
$$

$$
\begin{aligned}
(13)\quad &\nu_1(u)v^\alpha f(\overline{w}_m^\alpha) + \nu_2(u)f(\overline{w}_m^\alpha)v^\alpha\\
&- \nu_1(v)u^\alpha f(\overline{w}_m^\alpha) - \nu_2(v)f(\overline{w}_m^\alpha)u^\alpha\\
&= \sum_{i=1}^{m} f(w_1^\alpha,\dots,\{\nu_1(u)v^\alpha w_i^\alpha + \nu_2(u)w_i^\alpha v^\alpha - \nu_1(v)u^\alpha w_i^\alpha\\
&- \nu_2(v)w_i^\alpha u^\alpha + \nu_5(w_i)u^\alpha v^\alpha - \nu_5(w_i)v^\alpha u^\alpha\},w_{i+1}^\alpha,\dots,w_m^\alpha),
\end{aligned}
$$

$$(14)\ \mu_1(u,v)f(\overline{w}_m^\alpha) = \sum_{i=1}^{m} f(w_1^\alpha,\ldots,\{\mu_1(u,v)w_i^\alpha + \mu_2(u,w_i)v^\alpha - \mu_2(v,w_i)u^\alpha\},w_{i+1}^\alpha,\ldots,w_m^\alpha)$$

for all $u,v,w_1,\ldots,w_m \in \mathcal{S}$.

Note that the coefficient of $u^\alpha w_1^\alpha v^\alpha w_2^\alpha \ldots w_m^\alpha$ in (12) is equal to λ_2. But then $\lambda_2 = 0$ by Theorem 2.5. Similarly, computing coefficients at $u^\alpha v^\alpha w_2^\alpha \cdots w_m^\alpha$ and $w_1^\alpha u^\alpha w_2^\alpha \cdots w_m^\alpha$ in (13), we obtain $\nu_5 = 0$ and $\nu_1 = -\nu_2$, and computing coefficients at $u^\alpha w_2^\alpha \cdots w_m^\alpha$ and $v^\alpha w_2^\alpha \cdots w_m^\alpha$ in (14), we obtain $\mu_2 = 0$.

By (10) we now have

$$\begin{aligned}(\lambda_1+\lambda_5)(&u^\alpha v^\alpha w^\alpha + v^\alpha w^\alpha u^\alpha + w^\alpha u^\alpha v^\alpha \\ &- v^\alpha u^\alpha w^\alpha - u^\alpha w^\alpha v^\alpha - w^\alpha v^\alpha u^\alpha) \\ &+ 2\nu_1(u)v^\alpha w^\alpha + 2\nu_1(v)w^\alpha u^\alpha + 2\nu_1(u)v^\alpha w^\alpha \\ &- 2\nu_1(v)u^\alpha w^\alpha - 2\nu_1(u)w^\alpha v^\alpha - 2\nu_1(w)v^\alpha u^\alpha \\ &+ \mu_1(u,v)w^\alpha + \mu_1(v,w)u^\alpha + \mu_1(w,u)v^\alpha \\ &+ \omega(u,v,w) + \omega(v,w,u) + \omega(w,u,v) = 0,\end{aligned}$$

for all $u,v,w \in \mathcal{S}$. By Theorem 2.5 $\lambda_1 = -\lambda_5$, $\mu_1 = 0$, and since $\mathcal{Q}$ is 2-torsion free, $\nu_1 = 0$. The proof is complete. ∎

From now on we assume that $\mathcal{S}$ is a Jordan subring of $\mathcal{B}$ and note that, in view of (7), $[[\mathcal{S},\mathcal{S}],\mathcal{S}] \subseteq \mathcal{S}$. Thus, as a special case of Proposition 3.1, we have

Corollary 3.2: *Suppose that $\mathcal{S}$ is a Jordan subring of $\mathcal{B}$. If $\mathcal{S}^\alpha$ is a $(3m)$-free subset of $\mathcal{Q}$, then there exist $\lambda_1 \in \mathcal{C}$ and a map $\omega\colon \mathcal{S}^3 \to \mathcal{C}$ such that*

$$(15)\qquad [[u,v],w]^\alpha = \lambda_1[[u^\alpha,v^\alpha],w^\alpha] + \omega(u,v,w) \quad \textit{for all } u,v,w \in \mathcal{S}.$$

Our next goal is to find reasonable conditions under which the element λ_1 from Corollary3.2 is invertible. To this end we shall start by imposing the condition:

(16) Every element of $\mathcal{C}$ is either invertible or has square zero.

The reason for this condition will become clearer when f-derivations are studied in Section 4. At any rate it will turn out that condition (16) is enough to insure that λ_1 is invertible when f is a Jordan polynomial, that is, an element in $\mathcal{Z}\langle\mathcal{X}\rangle$ that can be expressed from the indeterminates by means of the sum and the Jordan product. In order to prove this, we shall need the following simple, but crucial lemma. First we introduce some further notation. By J we denote the free special Jordan algebra over $\mathcal{Z}$, i.e. the Jordan algebra of Jordan polynomials in $\mathcal{Z}\langle\mathcal{X}\rangle$, and by U the additive subgroup of $\mathcal{Z}\langle\mathcal{X}\rangle$ generated by $[[J,J],J]$. Of course, $U \subseteq J$ by (7).

LEMMA 3.3: *If $h(x_1,\ldots,x_m) \in J$ is multilinear of degree $m \geq 1$, then*

$$h(x_1 \ldots, x_m) = q(x_1,\ldots,x_{m-1}) \circ x_m + g(x_1,\ldots,x_m)$$

where $q \in J$ and $g \in U$.

Proof: Clearly, it suffices to consider the case when h is a Jordan monomial. We claim that any Jordan monomial h can be written as

$$h = ((((x_m \circ q_1) \circ q_2) \cdots) \circ q_{k-1}) \circ q_k$$

where the q_i's are Jordan monomials. We prove this by induction on the degree of h. If the degree is 1, there is nothing to prove. In general, we write $h = h_1 \circ h_2$, where h_1 and h_2 are Jordan monomials, both having degrees smaller than h. Since the Jordan product is commutative, there is no loss of generality in assuming that h_1 involves the indeterminate x_m. Apply the induction assumption on h_1 and our claim is proved.

Now, we prove the lemma by induction on k. For $k = 1$ the lemma is trivial, so let $k > 1$. By (7) it follows that

$$\begin{aligned} h &= (q_{k-1} \circ [(((x_m \circ q_1) \circ q_2) \cdots) \circ q_{k-2}]) \circ q_k \\ &\in (q_{k-1} \circ q_k) \circ [(((x_m \circ q_1) \circ q_2) \cdots) \circ q_{k-2}] + U, \end{aligned}$$

that is, $h \in ((((x_m \circ q_1) \circ q_2) \cdots) \circ q_{k-2}) \circ q'_{k-1} + U$, where $q'_{k-1} = q_{k-1} \circ q_k$. Apply the induction assumption and the desired conclusion follows. ∎

We assume henceforward that the conditions of Corollary 3.2 are fulfilled. We shall consider the following two conditions:

(i) α is one-to-one, and S does not satisfy a PI of degree $\leq m+4$.

(ii) f is a Jordan polynomial.

Let $f^{(i)}$ be the partial derivative of the polynomial f with respect to x_i (roughly speaking, one gets $f^{(i)}$ by replacing x_i in $f(x_1,\ldots,x_m)$ by 1).

LEMMA 3.4: *If (i) or (ii) holds, then λ_1 is invertible.*

Proof: Assuming that λ_1 is not invertible, we have $\lambda_1^2 = 0$ in view of (16). Now let $u, v, w, s, t \in \mathcal{S}$. It follows from (15) that

$$\begin{aligned}[[[[u,v],w],s],t]^\alpha &= \lambda_1[[[[u,v],w]^\alpha, s^\alpha], t^\alpha] + \omega([[u,v],w],s,t)\\ &= \lambda_1[[\lambda_1[[u^\alpha,v^\alpha],w^\alpha] + \omega(u,v,w), s^\alpha], t^\alpha] + \omega([[u,v],w],s,t)\\ &= \omega([[u,v],w],s,t).\end{aligned}$$

Thus, we have shown that

$$(17)\qquad [[r,s],t]^\alpha = \omega(r,s,t) \in \mathcal{C} \quad \text{for all } r \in [[\mathcal{S},\mathcal{S}],\mathcal{S}], s,t \in \mathcal{S}.$$

Assume now that (i) holds. Then, using (17) we write for $u \in [[\mathcal{S},\mathcal{S}],\mathcal{S}]$, $v \in \mathcal{S}$, $\overline{w}_m \in \mathcal{S}^m$,

$$(18)\qquad \begin{aligned}\omega(u,v,f(\overline{w}_m)) &= [[u,v], f(\overline{w}_m)]^\alpha\\ &= \sum_{i=1}^m f(w_1,\ldots,w_{i-1},[[u,v],w_i],w_{i+1},\ldots,w_m)^\alpha\\ &= \sum_{i=1}^m f(w_1^\alpha,\ldots,w_{i-1}^\alpha,[[u,v],w_i]^\alpha,\ldots,w_m^\alpha)\\ &= \sum_{i=1}^m f(w_1^\alpha,\ldots,w_{i-1}^\alpha,\omega(u,v,w_i),w_{i+1}^\alpha,\ldots,w_m^\alpha)\\ &= \sum_{i=1}^m \omega(u,v,w_i) f(w_1^\alpha,\ldots,w_{i-1}^\alpha,1,w_{i+1}^\alpha,\ldots,w_m^\alpha).\end{aligned}$$

It now follows from Theorem 2.5 that $\omega(u,v,f(\overline{w}_m)) = 0$ for each $u \in [[\mathcal{S},\mathcal{S}],\mathcal{S}]$, $v \in \mathcal{S}$, $\overline{w}_m \in \mathcal{S}^m$. Therefore $[[u,v],f(\overline{w}_m)]^\alpha = 0$, whence $[[u,v],f(\overline{w}_m)] = 0$ for all $u \in [[\mathcal{S},\mathcal{S}],\mathcal{S}]$, $v \in \mathcal{S}$, $\overline{w}_m \in \mathcal{S}^m$. This is a contradiction to $\mathcal{S}$ does not satisfy a PI of degree $\leq m+4$.

Now suppose that (ii) holds. We may assume without loss of generality that the monomial $x_1x_2\cdots x_m$ is involved in f with coefficient 1. Setting

$$\lambda = \begin{cases} \lambda_1 & \text{if } \lambda_1 \neq 0,\\ 1 & \text{if } \lambda_1 = 0,\end{cases}$$

we conclude from (15) that

$$(19)\qquad \lambda[[\mathcal{S},\mathcal{S}],\mathcal{S}]^\alpha \subseteq \mathcal{C}.$$

According to Lemma 3.3 we have

$$f(\overline{x}_m) = h(\overline{x}_{m-1}) \circ x_m + g(\overline{x}_m),$$

where $h \in J$ and $g \in U$. For $\overline{u}_m \in \mathcal{S}^m$ we set

$$\tau(\overline{u}_m) = \lambda g(\overline{u}_m)^\alpha$$

and note that $\tau(\overline{u}_m) \in \mathcal{C}$ by (19).

Clearly

$$f(\overline{u}_m^\alpha) = \{u_m \circ h(\overline{u}_{m-1})\}^\alpha + g(\overline{u}_m)^\alpha \tag{20}$$

for all $\overline{u}_m \in \mathcal{S}^m$. Using (20) we now get

$$\begin{aligned}\lambda f(\overline{u}_{m-1}^\alpha, f(\overline{v}_{m-1}^\alpha, u_m^\alpha)) &= \lambda f(\overline{u}_{m-1}, f(\overline{v}_{m-1}, u_m))^\alpha \\ &= \lambda f(\overline{u}_{m-1}, h(\overline{v}_{m-1}) \circ u_m + g(\overline{v}_{m-1}, u_m))^\alpha \\ &= \lambda f(\overline{u}_{m-1}^\alpha, \{h(\overline{v}_{m-1}) \circ u_m\}^\alpha + g(\overline{v}_{m-1}, u_m)^\alpha) \\ &= \lambda f(\overline{u}_{m-1}^\alpha, \{h(\overline{v}_{m-1}) \circ u_m\}^\alpha) + \tau(\overline{v}_{m-1}, u_m) f^{(m)}(\overline{u}_m^\alpha) \\ &= \lambda\{h(\overline{u}_{m-1}) \circ (h(\overline{v}_{m-1}) \circ u_m)\}^\alpha + \tau(\overline{u}_{m-1}, h(\overline{v}_{m-1}) \circ u_m) \\ &\quad + \tau(\overline{v}_{m-1}, u_m) f^{(m)}(\overline{u}_m^\alpha).\end{aligned} \tag{21}$$

Analogously, we have

$$\begin{aligned}\lambda f(\overline{v}_{m-1}^\alpha, f(\overline{u}_m^\alpha)) &= \lambda f(\overline{v}_{m-1}, f(\overline{u}_m))^\alpha \\ &= \lambda f(\overline{v}_{m-1}, u_m \circ h(\overline{u}_{m-1}) + g(\overline{u}_m))^\alpha \\ &= \lambda f(\overline{v}_{m-1}^\alpha, \{u_m \circ h(\overline{u}_{m-1})\}^\alpha) + \tau(\overline{u}_m) f^{(m)}(\overline{v}_m^\alpha) \\ &= \lambda\{h(\overline{v}_{m-1}) \circ (h(\overline{u}_{m-1}) \circ u_m)\}^\alpha + \tau(\overline{v}_{m-1}, u_m \circ h(\overline{u}_{m-1})) \\ &\quad + \tau(\overline{u}_m) f^{(m)}(\overline{v}_m^\alpha).\end{aligned} \tag{22}$$

It follows from (7) and (19) that

$$\begin{aligned}\lambda\{(h(\overline{v}_{m-1}) \circ u_m) \circ h(\overline{u}_{m-1})\}^\alpha - \lambda\{(h(\overline{u}_{m-1}) \circ u_m) \circ h(\overline{v}_{m-1})\}^\alpha \\ = \lambda[[h(\overline{u}_{m-1}), h(\overline{v}_{m-1})], u_m]^\alpha \in \mathcal{C}.\end{aligned}$$

Comparing (22) and (21), we obtain

$$\begin{aligned}\lambda f(\overline{u}_{m-1}^\alpha, f(\overline{v}_{m-1}^\alpha, u_m^\alpha)) - \lambda f(\overline{v}_{m-1}^\alpha, f(\overline{u}_m^\alpha)) \\ - \tau(\overline{v}_{m-1}, u_m) f^{(m)}(\overline{u}_m^\alpha) + \tau(\overline{u}_m) f^{(m)}(\overline{v}_m^\alpha) \in \mathcal{C}\end{aligned}$$

for all $\overline{u}_m \in \mathcal{S}^m$, $\overline{v}_{m-1} \in \mathcal{S}^{m-1}$. By Theorem 2.5,

$$\lambda f(\overline{x}_{m-1}, f(\overline{y}_{m-1}, x_m)) - \lambda f(\overline{y}_{m-1}, f(\overline{x}_m)) = 0 \tag{23}$$

as an element of $\mathcal{Z}\langle x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_{m-1}\rangle$. However, since $x_1 x_2 \cdots x_m$ is a monomial of f, $\lambda x_1 x_2 \cdots x_{m-1} y_1 y_2 \cdots y_{m-1} x_m$ should be a monomial of the polynomial on the left side of (23). With this contradiction the lemma is proved. ∎

LEMMA 3.5: *If either (i) or (ii) holds, then there exist an invertible $\lambda_2 \in \mathcal{C}$, a $\mathcal{Z}$-module map $\gamma: \mathcal{S} \to \mathcal{C}$ and a bilinear map $\nu: \mathcal{S}^2 \to \mathcal{C}$ such that*

$$(u \circ v)^\alpha = \lambda_2 u^\alpha \circ v^\alpha + \gamma(u)v^\alpha + \gamma(v)u^\alpha + \nu(u,v) \tag{24}$$

for all $u, v \in \mathcal{S}$. Furthermore, $2\lambda_2\nu(u,v) + \gamma(u \circ v) = \gamma(u)\gamma(v)$ for all $u, v \in \mathcal{S}$.

Proof: Define $B: \mathcal{S}^2 \to \mathcal{Q}$ by $B(u,v) = (u \circ v)^\alpha$, $u, v \in \mathcal{S}$. Corollary 3.2 and Lemma 3.4 show that

$$[[u^\alpha, v^\alpha], w^\alpha] = \lambda_1^{-1}\{[[u,v],w]^\alpha + \omega(u,v,w)\} \tag{25}$$

for all $u, v, w \in \mathcal{S}$. Clearly $[u \circ v, w] + [v \circ w, u] + [w \circ u, v] = 0$. Applying α to

$$[[u \circ v, w] + [v \circ w, u] + [w \circ u, v], t] = 0$$

and making use of (25) we get

$$[[B(u,v), w^\alpha] + [B(w,u), v^\alpha] + [B(v,w), u^\alpha], t^\alpha] = 0$$

for all $u, v, w, t \in \mathcal{S}$. Since $\mathcal{S}^\alpha$ is in particular 5-free, we conclude from Theorem 2.6 that

$$B(u,v) = \lambda_2' u^\alpha v^\alpha + \lambda_2'' v^\alpha u^\alpha + \gamma_1(u)v^\alpha + \gamma_2(v)u^\alpha + \nu(u,v)$$

for some $\lambda_2', \lambda_2'' \in \mathcal{C}$, $\gamma_1, \gamma_2: \mathcal{S} \to \mathcal{C}$, $\nu: \mathcal{S}^2 \to \mathcal{C}$. Since $B(u,v) = B(v,u)$, Theorem 2.5 shows that $\lambda_2' = \lambda_2'' = \lambda_2$ and $\gamma_1 = \gamma_2 = \gamma$. Therefore

$$(u \circ v)^\alpha = B(u,v) = \lambda_2(u^\alpha \circ v^\alpha) + \gamma(u)v^\alpha + \gamma(v)u^\alpha + \nu(u,v). \tag{26}$$

Next we show that λ_2 is invertible. Suppose that $\lambda_2^2 = 0$ and set

$$\lambda = \begin{cases} \lambda_2 & \text{if } \lambda_2 \neq 0, \\ 1 & \text{if } \lambda_2 = 0. \end{cases}$$

Then it follows easily from repeated use of (26) that

$$\lambda\{(u \circ v) \circ w\}^\alpha = \zeta_1(v,w)u^\alpha + \zeta_2(u,w)v^\alpha + \zeta_3(u,v)w^\alpha + \eta_1(u,v,w)$$

for all $u, v, w \in S$, where $\zeta_1, \zeta_2, \zeta_3: \mathcal{S}^2 \to \mathcal{C}$ and $\eta_1: \mathcal{S}^3 \to \mathcal{C}$ are some maps. Therefore

$$\begin{aligned}\lambda[[w,v],u]^\alpha &= \lambda\{(u \circ v) \circ w - (u \circ w) \circ v\}^\alpha \\ &= \zeta_4(v,w)u^\alpha + \zeta_5(u,w)v^\alpha + \zeta_6(u,v)w^\alpha + \eta_2(u,v,w),\end{aligned}$$

where $\zeta_4, \zeta_5, \zeta_6\colon \mathcal{S}^2 \to \mathcal{C}$ and $\eta_2\colon \mathcal{S}^3 \to \mathcal{C}$. But on the other hand, $\lambda[[w,v],u]^\alpha = \lambda\lambda_1[[w^\alpha, v^\alpha], u^\alpha] + \lambda\omega(u,v,w)$. Since $\lambda\lambda_1$ is nonzero, this contradicts Theorem 2.5.

Finally, we show that $2\lambda_2\nu(u,v) + \gamma(u \circ v) = \gamma(u)\gamma(v)$. This is done by computing $[[w,v],u]^\alpha$ in two different ways and applying Theorem 2.5. On the one hand, by Corollary 3.2

$$[[w,v],u]^\alpha = \lambda_1[[w^\alpha, v^\alpha], u^\alpha] + \omega(w,v,u). \tag{27}$$

On the other hand, in view of (7) we have

$$[[w,v],u]^\alpha = ((u \circ v) \circ w)^\alpha - ((u \circ w) \circ v)^\alpha. \tag{28}$$

Expanding (28) by repeated applications of (26) we see in particular that the coefficient of w^α is $2\lambda_2\nu(u,v) + \gamma(u \circ v) - \gamma(u)\gamma(v)$ (we leave the straightforward details to the reader). In view of Theorem 2.5 a comparison with (27) then completes the proof. ∎

Following [13, p. 535] we now define a map $\beta\colon \mathcal{S} \to \mathcal{Q}$ by

$$u^\beta = \lambda_2 u^\alpha + \frac{1}{2}\gamma(u).$$

LEMMA 3.6: *β is a Jordan homomorphism of $\mathcal{Z}$-algebras.*

Proof: Let $u, v \in \mathcal{S}$. Then

$$\begin{aligned}(u \circ v)^\beta =& \lambda_2(u \circ v)^\alpha + \frac{1}{2}\gamma(u \circ v)\\ =& \lambda_2^2(u^\alpha \circ v^\alpha) + \lambda_2\gamma(u)v^\alpha\\ &+ \lambda_2\gamma(v)u^\alpha + \lambda_2\nu(u,v) + \frac{1}{2}\gamma(u \circ v),\end{aligned}$$

while

$$\begin{aligned}u^\beta \circ v^\beta &= (\lambda_2 u^\alpha + \frac{1}{2}\gamma(u)) \circ (\lambda_2 v^\alpha + \frac{1}{2}\gamma(v))\\ &= \lambda_2^2(u^\alpha \circ v^\alpha) + \lambda_2\gamma(u)v^\alpha + \lambda_2\gamma(v)u^\alpha + \frac{1}{2}\gamma(u)\gamma(v).\end{aligned}$$

By the last part of Lemma 3.5 it is immediate that $(u \circ v)^\beta = u^\beta \circ v^\beta$, and the proof is complete. ∎

Setting $\lambda = \lambda_2^{-1}$ and $\mu(u) = -\frac{1}{2}\lambda_2^{-1}\gamma(u)$, we thus have $u^\alpha = \lambda u^\beta + \mu(u)$. We now summarize all partial results into the following statement.

THEOREM 3.7: *Let $\mathcal{B}$ and $\mathcal{Q}$ be associative algebras over a commutative unital ring $\mathcal{Z}$. Suppose that $\mathcal{Q}$ is a unital algebra with $\frac{1}{2} \in \mathcal{Q}$ and that every element in its center $\mathcal{C}$ is either invertible or has square zero. Let $\mathcal{S}$ be Jordan subalgebra of the algebra $\mathcal{B}$ and let $f(x_1, \ldots, x_m) \in \mathcal{Z}\langle \mathcal{X} \rangle$, $m \geq 2$ be a proper multilinear polynomial such that $f(u_1, \ldots, u_m) \in \mathcal{S}$ for all $u_1, \ldots, u_m \in \mathcal{S}$. Further, suppose that $\alpha\colon \mathcal{S} \to \mathcal{Q}$ is a $\mathcal{Z}$-module map satisfying*

$$f(u_1, \ldots, u_m)^\alpha = f(u_1^\alpha, \ldots, u_m^\alpha)$$

for all $u_1, \ldots, u_m \in \mathcal{S}$. Assume that the range of α is a $(3m)$-free subset of $\mathcal{Q}$ and that at least one of the following two conditions holds:

(i) *α is one-to-one and $\mathcal{S}$ does not satisfy a PI of degree $\leq m+4$.*

(ii) *f is a Jordan polynomial.*

Then

$$u^\alpha = \lambda u^\beta + \mu(u),$$

for all $u \in \mathcal{S}$, where $\lambda \in \mathcal{C}$ is invertible, $\mu\colon \mathcal{S} \to \mathcal{C}$ is a $\mathcal{Z}$-module map and $\beta\colon \mathcal{S} \to \mathcal{Q}$ is a Jordan homomorphism of $\mathcal{Z}$-algebras.

As mentioned in the introduction, we note that Theorem 3.7 essentially reduces the problem of characterizing f-homomorphisms to that of analyzing Jordan homomorphisms. Under various conditions, notably those involving the Zelmanov polynomial [25, 26] it is known that a Jordan homomorphism is just the restriction of an ordinary homomorphism.

We choose, however, to focus on an important special case of Theorem 3.7, one in which the Zelmanov polynomial is not required. Indeed, in Theorem 3.9 (which is our final goal) we will be concerned with a generalization of the situation in which $\mathcal{S}$ is the Jordan ring of symmetric elements of a $\mathcal{Z}$-algebra $\mathcal{B}$ with involution. Theorem 1.1 illustrates Theorem 3.9 with a specific but important example in which the somewhat unnatural freeness conditions are avoided in the hypothesis.

First, however, in what may be of independent interest, we make a couple of general observations (Lemma 3.8) about Jordan subrings of associative rings.

LEMMA 3.8: *Let $\mathcal{B}$ be an associative algebra over a commutative unital ring $\mathcal{Z}$ with Jordan subalgebra $\mathcal{S}$. Assume that $[\mathcal{S},\mathcal{S}]\circ[\mathcal{S},\mathcal{S}] \subseteq \mathcal{S}$ and $\frac{1}{2} \in \mathcal{Z}$. Let $\langle \mathcal{S} \rangle$ be the associative subalgebra of $\mathcal{B}$ generated by $\mathcal{S}$. Then $\langle \mathcal{S} \rangle = \mathcal{S} + [\mathcal{S},\mathcal{S}] + [\mathcal{S},\mathcal{S}]\circ\mathcal{S}$ and $\mathcal{S} + [\mathcal{S},\mathcal{S}]$ is a Lie ideal of $\langle \mathcal{S} \rangle$ containing $[\langle \mathcal{S} \rangle, \langle \mathcal{S} \rangle]$.*

Proof: Set $\mathcal{T} = \mathcal{S} + [\mathcal{S},\mathcal{S}] + [\mathcal{S},\mathcal{S}] \circ \mathcal{S}$. Clearly $\mathcal{T} \subseteq \langle \mathcal{S} \rangle$. To prove the opposite inclusion, it is enough to show that $xs \in \mathcal{T}$ for all $s \in \mathcal{S}$ and $x \in \mathcal{T}$. If $x \in \mathcal{S}$, then $xs = \frac{1}{2}([x,s] + x \circ s) \in \mathcal{T}$. Next, if $x \in [\mathcal{S},\mathcal{S}]$, then we may assume without loss of generality that $x = [u,v]$ for some $u,v \in \mathcal{S}$. We now have $xs = [u,v]s = \frac{1}{2}([u,v] \circ s + [[u,v],s]) \in \mathcal{T}$ because $[[u,v],s] = (s \circ v) \circ u - (s \circ u) \circ v \in \mathcal{S}$ by (7). Further, if $x \in [\mathcal{S},\mathcal{S}] \circ \mathcal{S}$, we may assume that $x = [u,v] \circ w$ for some $u,v,w \in \mathcal{S}$. Therefore

$$xs = \frac{1}{2}([[u,v] \circ w, s] + ([u,v] \circ w) \circ s).$$

Since $[[u,v],s] \in \mathcal{S}$, it follows from our assumption that

$$[[u,v] \circ w, s] = [u,v] \circ [w,s] + [[u,v],s] \circ w \in \mathcal{S} \subseteq \mathcal{T}. \tag{29}$$

Making use of (7) we see that

$$([u,v] \circ w) \circ s = [u,v] \circ (w \circ s) + [[s,[u,v]],w].$$

Since $w \circ s \in \mathcal{S}$, $[u,v] \circ (w \circ s) \in \mathcal{T}$. As $[s,[u,v]] \in \mathcal{S}$ by (7), we conclude that $[[s,[u,v]],w] \in \mathcal{T}$. Therefore $([u,v] \circ w) \circ s \in \mathcal{T}$ and so $xs \in \mathcal{T}$ for all $x \in \mathcal{T}$, $s \in \mathcal{S}$. Thus $\mathcal{T} = \langle \mathcal{S} \rangle$.

Set $\mathcal{U} = \mathcal{S} + [\mathcal{S},\mathcal{S}]$. It is enough to show that $\mathcal{U} \supseteq [\langle \mathcal{S} \rangle, \langle \mathcal{S} \rangle]$. By [11, Lemma 9.1.2], $[\langle \mathcal{S} \rangle, \langle \mathcal{S} \rangle] = [\mathcal{S}, \langle \mathcal{S} \rangle]$. Note that $[\mathcal{S},\mathcal{S}] \subseteq \mathcal{U}$ while $[[\mathcal{S},\mathcal{S}],\mathcal{S}] \subseteq \mathcal{S} \subseteq \mathcal{U}$ by (7). Finally, $[\mathcal{S},[\mathcal{S},\mathcal{S}] \circ \mathcal{S}] \subseteq \mathcal{U}$ by (29). Thus $[\mathcal{S}, \langle \mathcal{S} \rangle] \subseteq \mathcal{U}$ and the proof is thereby complete.

THEOREM 3.9: *Let $\mathcal{B}$ and $\mathcal{Q}$ be associative algebras over a commutative unital ring $\mathcal{Z}$. Suppose that $\mathcal{Q}$ is a unital algebra with $\frac{1}{2} \in \mathcal{Q}$ and that every element in its center $\mathcal{C}$ is either invertible or has square zero and $\mathcal{C}$ is a direct summand of the $\mathcal{C}$-module $\mathcal{Q}$. Let $\mathcal{S}$ be a Jordan subalgebra of the algebra $\mathcal{B}$ such that $\mathcal{S} \cap [\mathcal{S},\mathcal{S}] = 0$ and $[\mathcal{S},\mathcal{S}] \circ [\mathcal{S},\mathcal{S}] \subseteq \mathcal{S}$. Let $f(x_1,\dots,x_m) \in \mathcal{Z}\langle \mathcal{X} \rangle$, $m \geq 2$ be a proper multilinear polynomial such that $f(u_1,\dots,u_m) \in \mathcal{S}$ for all $u_1,\dots,u_m \in \mathcal{S}$. Further, suppose that $\alpha\colon \mathcal{S} \to \mathcal{Q}$ is a $\mathcal{Z}$-module map satisfying*

$$f(u_1,\dots,u_m)^\alpha = f(u_1^\alpha,\dots,u_m^\alpha)$$

for all $u_1,\dots,u_m \in \mathcal{S}$. Assume that $\mathcal{S}^\alpha$ is a $\max\{3m,7\}$-free subset of $\mathcal{Q}$ and that at least one of the following two conditions holds:

(i) *α is one-to-one and $\mathcal{S}$ does not satisfy a PI of degree $\leq m+4$.*
(ii) *f is a Jordan polynomial.*

Then there exist $\lambda \in \mathcal{C}$, $\lambda^{m-1} = 1$, a $\mathcal{Z}$-module map $\psi\colon \mathcal{S} \to \mathcal{C}$, and a $\mathcal{Z}$-algebra homomorphism $\sigma\colon \langle \mathcal{S} \rangle \to \langle \mathcal{S}^\alpha \rangle \mathcal{C} + \mathcal{C}$ such that $u^\alpha = \lambda u^\sigma + \psi(u)$ for all $u \in \mathcal{S}$. Furthermore, if $f^{(i)} \neq 0$ for some i, then $\psi = 0$.

Proof: We may apply Theorem 3.7 to obtain

$$u^\alpha = \lambda u^\beta + \mu(u), \quad u \in \mathcal{S}, \tag{30}$$

where $\lambda \in \mathcal{C}$ is invertible, $\mu\colon \mathcal{S} \to \mathcal{C}$ is a $\mathcal{Z}$-module map, and $\beta\colon \mathcal{S} \to \mathcal{Q}$ is a Jordan homomorphism of $\mathcal{Z}$-algebras. For convenience we may assume $\mathcal{B} = \langle \mathcal{S} \rangle$. By Lemma 3.8, $\mathcal{U} = \mathcal{S} \oplus [\mathcal{S}, \mathcal{S}]$ is a Lie ideal of $\mathcal{B}$. We define a mapping $\gamma\colon \mathcal{U} \to \overline{\mathcal{Q}} = \mathcal{Q}/\mathcal{C}$ according to the rule

$$u + \sum [v_i, w_i] \to \overline{u^\beta + \sum [v_i^\beta, w_i^\beta]}.$$

To show that γ is well-defined suppose $\sum [v_i, w_i] = 0$. Then we have $\sum [[v_i, w_i], t] = 0$ for all $t \in \mathcal{S}$ and, applying β, we obtain

$$\sum [[v_i^\beta, w_i^\beta], t^\beta] = \sum [[v_i^\beta, w_i^\beta], \lambda^{-1} t^\alpha] = 0$$

for all $t \in \mathcal{S}$. Since $\mathcal{S}^\alpha$ is $3m$-free this puts $\sum [v_i^\beta, w_i^\beta] \in \mathcal{C}$, that is, $\overline{\sum [v_i^\beta, w_i^\beta]} = 0$. We now show that γ is a Lie homomorphisms. Let $x, y \in \mathcal{U}$. It is enough to show that $[x, y]^\gamma = [x^\gamma, y^\gamma]$. There are three cases to consider.

Case 1: Assume that $x, y \in \mathcal{S}$. Then

$$[x, y]^\gamma = \overline{[x^\beta, y^\beta]} = [\overline{x^\beta}, \overline{y^\beta}] = [x^\gamma, y^\gamma].$$

Case 2: Suppose that $x \in [\mathcal{S}, \mathcal{S}]$ and $y \in \mathcal{S}$. Recalling that γ is an additive map, we may assume without loss of generality that $x = [u, v]$ for some $u, v \in \mathcal{S}$. Since β is a Jordan homomorphism, (7) together with Case 1 yield

$$\begin{aligned}
[[u, v], y]^\gamma &= \{(y \circ v) \circ u - (y \circ u) \circ v\}^\gamma \\
&= \overline{\{(y \circ v) \circ u\}^\beta - \{(y \circ u) \circ v\}^\beta} \\
&= \overline{(y^\beta \circ v^\beta) \circ u^\beta - (y^\beta \circ u^\beta) \circ v^\beta} = \overline{[[u^\beta, v^\beta], y^\beta]} \\
&= [[\overline{u^\beta}, \overline{v^\beta}], \overline{y^\beta}] = [[u^\gamma, v^\gamma], y^\gamma] = [x^\gamma, y^\gamma].
\end{aligned}$$

Case 3: Assume $x, y \in [\mathcal{S}, \mathcal{S}]$. Again we may suppose that $x = [u, v]$ and $y = [p, q]$ for some $u, v, p, q \in \mathcal{S}$. In view of (7), $[[\mathcal{S}, \mathcal{S}], \mathcal{S}] \subseteq \mathcal{S}$ and so we get by the above cases that

$$\begin{aligned}
[x, y]^\gamma &= [[u, v], [p, q]]^\gamma = \{[[u, [p, q]], v] + [u, [v, [p, q]]]\}^\gamma \\
&= [[u, [p, q]]^\gamma, v^\gamma] + [u^\gamma, [v, [p, q]]^\gamma] \\
&= [[u^\gamma, [p^\gamma, q^\gamma]], v^\gamma] + [u^\gamma, [v^\gamma, [p^\gamma, q^\gamma]]] \\
&= [[u^\gamma, v^\gamma], [p^\gamma, q^\gamma]] = [x^\gamma, y^\gamma].
\end{aligned}$$

Therefore γ is a Lie homomorphism whose image $\mathcal{U}^\gamma$ can be written as $\overline{\mathcal{T}}$ where $\mathcal{T} = \lambda^{-1}\mathcal{S}^\alpha + \lambda^{-2}[\mathcal{S}^\alpha, \mathcal{S}^\alpha]$. Since $\mathcal{S}^\alpha$ is 7-free, by Theorem 2.3(b) $\lambda^{-1}\mathcal{S}^\alpha$ is 7-free and so $\mathcal{T}$ is 7-free by Theorem 2.3(a). Noting that 1 is the only nonzero idempotent in $\mathcal{C}$ we see by Theorem 2.7 that there exists a map $\sigma: \langle\mathcal{U}\rangle \to \langle\mathcal{S}^\alpha\rangle\mathcal{C}+\mathcal{C}$ such that σ is either a homomorphism or the negative of an antihomomorphism χ and such that $\overline{u^\sigma} = u^\gamma$ for all $u \in \mathcal{U}$. In particular, for $u \in \mathcal{S}$, we have $\overline{u^\sigma} = u^\gamma = \overline{u^\beta}$ and so $u^\sigma = u^\beta + \rho(u)$, $\rho: \mathcal{S} \to \mathcal{C}$. We claim that σ is a homomorphism. Indeed, suppose $\sigma = -\chi$. For $u, v \in \mathcal{S}$, on the one hand,

$$(u \circ v)^\sigma = -(u \circ v)^\chi = -u^\chi \circ v^\chi = -u^\sigma \circ v^\sigma. \tag{31}$$

On the other hand,

$$\begin{aligned}(u \circ v)^\sigma &= (u \circ v)^\beta + \rho(u \circ v) = u^\beta \circ v^\beta + \rho(u \circ v)\\ &= (u^\sigma - \rho(u)) \circ (v^\sigma - \rho(v)) + \rho(u \circ v)\\ &= u^\sigma \circ v^\sigma - 2\rho(u)v^\sigma - 2\rho(v)u^\sigma + 2\rho(u)\rho(v) + \rho(u \circ v).\end{aligned} \tag{32}$$

Comparing (31) and (32) and multiplying through by λ^{-2} we obtain

$$u^\alpha \circ v^\alpha + \tau_1(u,v)u^\alpha + \tau_2(u,v)v^\alpha + \tau_3(u,v) = 0$$

for all $u, v \in \mathcal{S}$ and appropriate $\tau_i: \mathcal{S} \to \mathcal{C}$. In view of Theorem 2.5 this contradicts the $3m$-freeness of $\mathcal{S}^\alpha$. Therefore σ is a homomorphism and, substituting $u^\beta = u^\sigma - \rho(u)$ in (30), we see that

$$u^\alpha = \lambda u^\sigma + \psi(u), \quad \psi: \mathcal{S} \to \mathcal{C} \tag{33}$$

for all $u \in \mathcal{S}$. Finally, using (33), we may now write, for all $\overline{u}_m \in \mathcal{S}^m$,

$$\begin{aligned}f(\overline{u}_m^\alpha) &= f(\overline{u}_m)^\alpha = \lambda f(\overline{u}_m)^\sigma + \psi(f(\overline{u}_m))\\ &= \lambda f(\overline{u}_m^\sigma) + \psi(f(\overline{u}_m))\\ &= \lambda f(\ldots, \lambda^{-1}u_i^\alpha - \psi(u_i), \ldots) + \psi(f(\overline{u}_m))\\ &= \lambda^{1-m} f(\overline{u}_m^\alpha) - \sum_{i=1}^{m} \lambda^{2-m}\psi(u_i) f(u_1^\alpha, \ldots, 1, \ldots, u_m^\alpha)\\ &\qquad + \text{lower powers.}\end{aligned} \tag{34}$$

By Theorem 2.5 we conclude from (34) that $\lambda^{m-1} = 1$ and, if some $f^{(i)} \neq 0$, that $\psi = 0$. The proof of Theorem 3.9 is now complete. ∎

Proof of Theorem 1.1: We proceed to show that the conditions of Theorem 3.9 are satisfied. First, it is well-known and easy to see that $\mathcal{S} \cap [\mathcal{S}, \mathcal{S}] = 0$ and $\mathcal{S}$

is closed under $[\ ,\ ]\circ[\ ,\ ]$. Furthermore, $\mathcal{Q}_{mr}(\mathcal{I}) = \mathcal{Q}$ [11, Proposition 2.1.10] and $\deg(\mathcal{I}) = \deg(\mathcal{A})$ by Theorem 2.2(a). Thus $\deg(\mathcal{I}) > \max\{6m+1, 15\}$. By Theorem 2.2(c) $\mathcal{S}(\mathcal{I})$ is a $\max\{3m,7\}$-free subset of $\mathcal{Q}$, and so by Theorem 2.3(a) $\mathcal{S}^\alpha$ is a $\max\{3m,7\}$-free subset of $\mathcal{Q}$. Therefore all the conditions of Theorem 3.9 hold and the proof is thereby complete. ∎

4. f-derivations

In an analogous fashion to the concept of an f-homomorphism we define the notion of an f-derivation in a natural way as follows. Let $\mathcal{Q}$ be a $\mathcal{Z}$-algebra with 1, and let $\mathcal{A}$ be a $\mathcal{Z}$-subalgebra of $\mathcal{Q}$. Let $f(x_1,\dots,x_m) \in \mathcal{Z}\langle\mathcal{X}\rangle$ be a proper multilinear polynomial of degree $m \geq 2$, and let $\mathcal{S}$ be a $\mathcal{Z}$-submodule of $\mathcal{A}$ such that $f(\overline{s}_m) \in \mathcal{S}$ for all $\overline{s}_m = (s_1,\dots,s_m) \in \mathcal{S}^m$. A $\mathcal{Z}$-module map $\delta\colon \mathcal{S} \to \mathcal{Q}$ is said to be an f-derivation if

$$f(\overline{s}_m)^\delta = \sum_{i=1}^{m} f(s_1,\dots,s_{i-1},s_i^\delta,\dots,s_m)$$

for all $\overline{s}_m \in \mathcal{S}^m$.

The goal of this section is to analyze f-derivations of $\mathcal{S}$ where $\mathcal{S}$ is a Jordan subalgebra of $\mathcal{A}$ and appropriate conditions hold on $\mathcal{A}$ and $\mathcal{S}$.

We now proceed to prove the analogues of Theorems 3.7 and 3.9, combining them in a single result.

THEOREM 4.1: *Let $\mathcal{S}$ be a Jordan subalgebra of a unital $\mathcal{Z}$-algebra $\mathcal{Q}$, whose center $\mathcal{C}$ is a field. Let $\frac{1}{2} \in \mathcal{C}$. Let $f(x_1,\dots,x_m) \in \mathcal{Z}\langle\mathcal{X}\rangle$, $m > 1$, be a proper multilinear polynomial. Suppose $\mathcal{S}$ is closed under f and is a $\max\{3m,7\}$-free subset of $\mathcal{Q}$. Let $\delta\colon \mathcal{S} \to \mathcal{Q}$ be an f-derivation. Then:*

(a) *There exists a Jordan derivation $\rho\colon \mathcal{S} \to \mathcal{Q}$, $\lambda \in \mathcal{C}$, and a $\mathcal{Z}$-module map $\mu\colon \mathcal{S} \to \mathcal{C}$ such that*

$$s^\delta = s^\rho + \lambda s + \mu(s), \quad s \in \mathcal{S}.$$

(b) *If $\mathcal{S} \cap [\mathcal{S},\mathcal{S}] = 0$ and $[\mathcal{S},\mathcal{S}] \circ [\mathcal{S},\mathcal{S}] \subseteq \mathcal{S}$, then there exists a derivation $d\colon \langle\mathcal{S}\rangle \to \mathcal{Q}$, $\lambda \in \mathcal{C}$, and a $\mathcal{Z}$-module map $\mu\colon \mathcal{S} \to \mathcal{C}$ such that*

$$s^\delta = s^d + \lambda s + \mu(s), \quad s \in \mathcal{S}.$$

Furthermore, we have:

(b1) *If the characteristic of $\mathcal{C}$ does not divide $m-1$ then $\lambda = 0$.*

(b2) *If $f^{(i)} \neq 0$ for some i then $\mu = 0$.*

Proof: We form the ring $\widehat{\mathcal{Q}} = \mathcal{Q} \oplus \mathcal{Q}$ with multiplication given by $(s,t)(u,v) = (su, sv+tu)$, $s,t,u,v \in \mathcal{Q}$. (Note that $\widehat{\mathcal{Q}}$ is really just the ring of 2×2 matrices over $\mathcal{Q}$ of the form $s(e_{11}+e_{22})+te_{12}$.) The center of $\widehat{\mathcal{Q}}$ is $\widehat{\mathcal{C}} = \mathcal{C}\oplus\mathcal{C}$ and satisfies condition (16): every element of $\widehat{\mathcal{C}}$ is either invertible or has square zero. We define $\alpha\colon \mathcal{S} \to \widehat{\mathcal{Q}}$ according to the rule $s \to (s, s^\delta)$ and it is easily seen that α is a one-to-one f-homomorphism.

The fact that $\mathcal{S}$ is $\max\{3m,7\}$-free implies that $\mathcal{S}$ does not satisfy a PI of degree $\leq m+4$, and so condition (i) is satisfied.

Furthermore, $\mathcal{S}^\alpha$ is a $\max\{3m,7\}$-free subset of $\widehat{\mathcal{Q}}$ by Theorem 2.4, and so we may apply (an equivalent form of) Theorem 3.7 to the map α (with $\widehat{\mathcal{Q}}$ now playing the role of $\mathcal{Q}$) to conclude that there exist a Jordan homomorphism $\beta\colon \mathcal{S} \to \widehat{\mathcal{Q}}$, an invertible element $c \in \widehat{\mathcal{C}}$ and a $\mathcal{Z}$-module map $\gamma\colon \mathcal{S} \to \widehat{\mathcal{C}}$ such that

$$s^\beta = cs^\alpha + \gamma(s) \quad \text{for all } s \in \mathcal{S}. \tag{35}$$

We may write $c = (\tau, \lambda')$, τ, $\lambda' \in \mathcal{C}$, $\tau \neq 0$, and $\gamma(s) = (\chi(s), \mu'(s))$ where χ, μ' are $\mathcal{Z}$-module maps of $\mathcal{S}$ into $\mathcal{C}$. Comparing the first components of $(s\circ t)^\beta = s^\beta \circ t^\beta$, $s,t \in \mathcal{S}$, we see that

$$(\tau^2 - \tau)(s\circ t) + 2\tau\chi(t)s + 2\tau\chi(s)t + 2\chi(s)\chi(t) - \chi(s\circ t) = 0.$$

Since $\mathcal{S}$ is $3m$-free, by Theorem 2.5 we conclude that $\tau = 1$ and $\chi = 0$. Thus (35) reduces to $s^\beta = (s, s^\delta + \lambda' s + \mu'(s))$ for all $s \in \mathcal{S}$. Accordingly we set $s^\rho = s^\delta + \lambda' s + \mu'(s)$, for all $s \in \mathcal{S}$, and note (from the fact that β is a Jordan homomorphism) that ρ is a Jordan derivation. This completes the proof of (a).

We now turn our attention to the proof of (b). Since all the conditions of Theorem 3.9 are satisfied, we conclude that there exist a homomorphism $\varphi\colon \langle\mathcal{S}\rangle \to \widehat{\mathcal{Q}}$, an element $c \in \widehat{\mathcal{C}}$ with $c^{m-1} = 1$, and a $\mathcal{Z}$-module map $\gamma\colon \mathcal{S} \to \widehat{\mathcal{C}}$ such that

$$s^\varphi = cs^\alpha + \gamma(s), \quad s \in \mathcal{S}. \tag{36}$$

We write $c = (\tau, \lambda')$, $\tau, \lambda' \in \mathcal{C}$, noting from $c^{m-1} = 1$ that in particular

$$(m-1)\tau^{m-2}\lambda' = 0. \tag{37}$$

We also write $\gamma(s) = (\chi(s), \mu'(s))$ where χ, μ' are $\mathcal{Z}$-module maps of $\mathcal{S}$ into $\mathcal{C}$. Since φ is also a Jordan homomorphism, (36) can be used in place of (35) and accordingly we may again conclude that $\tau = 1$ and $\chi = 0$. Therefore we have

$$s^{\varphi} = (s, s^{\delta} + \lambda' s + \mu'(s)), \quad s \in \mathcal{S}. \tag{38}$$

It follows from (38) that for all $x \in \langle \mathcal{S} \rangle$, $x^{\varphi} = (x, x^{d})$, where $d\colon \langle \mathcal{S} \rangle \to \mathcal{Q}$ is a well-defined $\mathcal{Z}$-module map. Since $(xy)^{\varphi} = x^{\varphi} y^{\varphi}$, $x, y \in \langle \mathcal{S} \rangle$, we see that $(xy)^{d} = xy^{d} + x^{d}y$, that is, d is a derivation. In particular, in view of (38), $s^{d} = s^{\delta} + \lambda' s + \mu'(s)$, $s \in \mathcal{S}$, and the first part of (b) has been proved. If the characteristic of $\mathcal{C}$ does not divide $m - 1$ it follows from (37) that $\lambda' = 0$, thus establishing (b1). If $f^{(i)}$ is not equal to 0 for some i, it follows from Theorem 3.9 that $\gamma = 0$ and in particular that $\mu' = 0$. Thus (b2) is proved and with it the proof of Theorem 4.1 is now complete. ∎

To illustrate (b1) in Theorem 4.1 we first remark that if $\mathcal{Q}$ is $(m-1)$-torsion and λ is any element of $\mathcal{C}$ then the map δ given by $s^{\delta} = \lambda s$ is an f-derivation.

Next we illustrate (b2) as follows. Let $\mathcal{C}$ be a field, let $\mathcal{A}$ be the free noncommutative algebra in indeterminates $y_1, y_2, \ldots, y_{2m}$ over $\mathcal{C}$ with constant term 0, let $*$ be the "reversal" operator on $\mathcal{A}$, and let $\mathcal{S}$ be the symmetric elements of $\mathcal{A}$ under the involution $*$. Clearly, the set of all elements of the form $M_k = y_{i_1} \cdots y_{i_k} + y_{i_k} \cdots y_{i_1}$ constitutes a $\mathcal{C}$-basis for $\mathcal{S}$. Let $\mu\colon \mathcal{S} \to \mathcal{C}$ be a linear map subject to the requirement that $\mu(M_k) = 0$ for $k \geq 2m$. We define δ by $s^{\delta} = \mu(s)$, $s \in \mathcal{S}$ and let

$$f(x_1, \ldots, x_{2m}) = St_{2m}(x_1, \ldots, x_{2m})$$

where St_{2m} is the standard polynomial of degree $2m$.

Clearly $f(\overline{s}_{2m})^{\delta} = 0$ for all $\overline{s}_{2m} \in \mathcal{S}^{2m}$. On the other hand, we see that

$$\sum_{i=1}^{2m} f(s_1, \ldots, \mu(s_i), \ldots, s_{2m}) = \sum_{i=1}^{2m} \mu(s_i) f(s_1, \ldots, 1, \ldots, s_{2m}) = 0.$$

Therefore δ is an f-derivation.

Proof of Theorem 1.2: By Theorem 2.2 all conditions of Theorem 4.1 are satisfied.

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